QUOTIENT UNIFORMITIES

by

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THESIS

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ABSTRACT

In this paper the notion of a quotient topology is extended to uniform spaces, and a quotient uniformity is defined for a uniform space. After the definition is validated its basic properties are investigated and its relation to the quotient topology is discussed.

TABLE OF CONTENTS

I. INTRODUCTION

In this paper the notion of a quotient uniformity is discussed. The idea was largely motivated by the following question. Suppose a quotient uniformity can be defined, then is the topology induced by it the quotient topology? In general, the answer to this question is negative, but in Chapter 5 an important sufficient condition for equality is established.

For understanding the paper a knowledge of general topology is expected, but for convenience to the reader and to avoid ambiguity, the more important terms are defined here. In the following X refers to a topological space. X is said to satisfy the _first axiom of countability_ if the neighborhood system of each point has a countable base. X satisfies the _second axiom of countability_ if it has a countable base. X is said to be a T_1 _space_ if every singleton is closed. X is _Hausdorff_ if points can be separated by disjoint open sets. X is _compact_ if every open cover has a finite subcover. Finally X is said to be _completely regular_ if for every point in X and every closed set not containing the point there is a continuous real valued function which is 1 on the point and 0 on the closed set. Caution: "Completely regular" as used in this paper _does not_ assume the T_1 separation axiom.

If f is a function from X to a topological space Y then f is _open_ if the image of each open set in X is open in Y. Similarly f is _closed_ if the image of every closed set in X is closed in Y.

The complement of a set A is denoted by ~A.

II. THE CONCEPT OF A QUOTIENT UNIFORMITY

In this chapter the notion of a quotient uniformity is defined by analogy with the concept of a quotient topology. It is known that given a topological space $(X,\mathcal{T})$ and a function of f from X onto an arbitrary set Y it is possible to define a topology on Y in the following way:

$$\mathcal{T}(f) = \{ G : G \subset Y,\ f^{-1}(G) \in \mathcal{T} \}$$

The topology $\mathcal{T}(f)$ is called the "quotient topology with respect to f" and it has the property that it is the largest topology on Y which makes f a continuous function.

It is natural to conjecture that a similar construction relative to a uniform space $(X,\mathcal{U})$ and an arbitrary set Y will result in a uniformity $\mathcal{U}(f)$ defined on the set Y. In order to make the statement of this problem more precise the following definitions are needed:

Definition 2.1: Let X be a set, and let $X \times X$ denote the cartesian product of X with itself. The collection of all pairs (x,x) is called the diagonal and is denoted by $\Delta(X)$.

Let U be contained in $X \times X$ then U^{-1} is the set of all pairs (x,y) such that (y,x) is in U. If U and V are contained in $X \times X$ then $U \circ V$ is defined to be the set of all (x,z) such that there is a y in X with (x,y) in V and (y,z) in U. Finally, U[G], for G contained in X, is the collection of all y such that (x,y) is in U for some x in G.

Definition 2.2: A uniformity for a set X is a non-void family of subsets of $X \times X$ satisfying the following properties:

(a) for each U in $\mathcal{U}$, $\Delta(X)$ is contained in U;

(b) if U is in $\mathcal{U}$ then U^{-1} is also in $\mathcal{U}$;

(c) if U is in $\mathcal{U}$ then there is a V in $\mathcal{U}$ such that $V \circ V$ is contained in U;

(d) if U and V are in $\mathcal{U}$ then their intersection is again in $\mathcal{U}$;

(e) if U is in $\mathcal{U}$ and V contains U then V is in $\mathcal{U}$.

<u>Definition 2.3</u>: if X is a set and $\mathcal{U}$ a uniformity for X, then the pair $(X,\mathcal{U})$ is called a <u>uniform space</u>.

Suppose f maps X into Y then the induced function <u>f_2</u> from $X \times X$ into $Y \times Y$ is defined by $f_2(x,y) = (f(x),f(y))$.

Now suppose f is a function from a uniform space $(X,\mathcal{U})$ onto a set Y. Define:

$$\mathcal{V} = \{V : V \subset Y \times Y \text{ and } f_2^{-1}(V) \in \mathcal{U}\} \quad .$$

For a proof that $\mathcal{V}$ is a uniformity on Y the following lemmas are needed.

<u>Lemma 2.1</u>: Let f be a function from X into Y and let $V \subset Y \times Y$; then $(f_2^{-1}(V))^{-1} = f_2^{-1}(V^{-1})$.

<u>Proof</u>: Let (x,y) be in $(f_2^{-1}(V))^{-1}$, then (y,x) is in $f_2^{-1}(V)$, hence $(f(y),f(x))$ is in V. This implies that $(f(x),f(y))$ is in V^{-1} and so (x,y) is in $f_2^{-1}(V^{-1})$. Similarly, let (x,y) be in $f_2^{-1}(V^{-1})$; then $(f(x),f(y))$ is in V^{-1}, thus $(f(y),f(x))$ is in V. This implies that (y,x) is in $f_2^{-1}(V)$ and so (x,y) is in $(f_2^{-1}(V))^{-1}$.

<u>Lemma 2.2</u>: Let f be a function from X into Y and let $V \subset X \times X$ then $f_2(V \circ V) \subset f_2(V) \circ f_2(V)$.

<u>Proof</u>: Let (x,y) be in $f_2(V\ V)$. Then there are elements a and

b in X such that $f(a) = x$, $f(b) = y$ and (a,b) is in $V \circ V$. Thus there is a c in X such that (a,c) and (c,b) are in V. It follows that $(f(a),f(c)) = (x,f(c))$ and $(f(c),f(b)) = (f(c),y)$ are both in $f_2(V)$ hence (x,y) is in $f_2(V) \circ f_2(V)$.

Theorem 2.1: Let f be a function from a uniform space $(X,\mathcal{U})$ onto a set Y. Then the collection $\mathcal{V}$ is a uniformity on Y.

Proof: Let V be in $\mathcal{V}$ and let y be in Y; then $f_2^{-1}(V)$ is in $\mathcal{U}$, and there is an x in X such that $f(x) = y$. We know (x,x) is in $f_2^{-1}(V)$, hence $(f(x),f(x)) = (y,y)$ is in $f_2(f_2^{-1}(V)) = V$. Thus $\Delta(Y) \subset V$ for all V in $\mathcal{V}$. Let $V \in \mathcal{V}$; then $f_2^{-1}(V)$ is in $\mathcal{U}$ thus $(f_2^{-1}(V))^{-1} = f_2^{-1}(V^{-1})$ is in $\mathcal{U}$. Hence V^{-1} is in $\mathcal{V}$. Moreover, $f_2^{-1}(V)$ in $\mathcal{U}$ implies that there is a Z in $\mathcal{U}$ such that $Z \circ Z \subset f_2^{-1}(V)$. Now $f_2(Z \circ Z) \subset f_2(Z) \circ f_2(Z) \subset f_2(f_2^{-1}(V)) = V$, and $f_2(Z)$ is in $\mathcal{V}$ since $f_2^{-1}(f_2(Z)) \supset Z$. Let V_1, V_2 be in $\mathcal{V}$; then $f_2^{-1}(V_1)$, $f_2^{-1}(V_2)$ are both in $\mathcal{U}$, thus $f_2^{-1}(V_1) \cap f_2^{-1}(V_2) = f_2^{-1}(V_1 \cap V_2)$ is in $\mathcal{U}$ and it follows that $V_1 \cap V_2$ is in $\mathcal{V}$. Finally let V be in $\mathcal{V}$ and suppose V_1 contains V. We know $f_2^{-1}(V)$ is in $\mathcal{U}$, and since $f_2^{-1}(V) \subset f_2^{-1}(V_1)$ it follows that $f_2^{-1}(V_1)$ is in $\mathcal{U}$, which implies that V_1 is in $\mathcal{V}$. Thus properties (a)-(e) are satisfied and hence $\mathcal{V}$ is a uniformity on Y.

Thus the generalization of the concept of quotient topology to uniform spaces is a valid one, and with this similarity in mind we call the uniformity of Theorem 1.1 the "quotient uniformity with respect to "f" and denote it by $\mathcal{U}(f)$.

It turns out that there is an alternate way of describing the quotient uniformity which in certain instances is more useful.

Theorem 2.2: Let f be a function from a uniform space $(X,\mathcal{U})$ onto a set Y such that Y has the quotient uniformity $\mathcal{U}(f)$. Then,

$$\mathcal{U}(f) = \{ f_2(U): U \in \mathcal{U} \}.$$

Proof: Let V be in $\mathcal{U}(f)$; then $f_2^{-1}(V)$ is in $\mathcal{U}$. Letting $\bar{U} = f_2^{-1}(V)$, we have $f_2(\bar{U}) = f_2(f_2^{-1}(V)) = V$. Thus V is in $\{f_2(U): U \in \mathcal{U}\}$. Similarly let V be in $\{f_2(U): U \in \mathcal{U}\}$; then $V = f_2(U)$ for some U in $\mathcal{U}$. Further $f_2^{-1}(V) = f_2^{-1}(f_2(U)) \supset U$, hence $f_2^{-1}(V)$ is in $\mathcal{U}$, which implies that V is in $\mathcal{U}(f)$.

It is worth noting that this description has no anology in the theory of quotient topologies.

Theorem 2.1 provides a means of generating new uniformities. In fact, for each function f from $(X,\mathcal{U})$ onto a set Y there is an associated uniformity on Y. A few illustrative examples follow.

Example 2.1: Let $(X,\mathcal{U})$ be a uniform space where $\mathcal{U}$ is the indiscrete uniformity (i.e., consisting only of the set $X \times X$). Then by virtue of Theorem 2.2 for any onto function f the associated uniformity on Y is also indiscrete.

Example 2.2: Similarly if $\mathcal{U}$ is the discrete uniformity (consists of all sets which contain the diagonal) then for any function f, the associated uniformity on Y is also discrete. This follows since $f_2(\Delta(X)) = \Delta(Y)$, hence $\Delta(Y)$ is in $\mathcal{U}(f)$ which implies by property (e) of a uniformity that $\mathcal{U}(f)$ is discrete.

Example 2.3: Let $X = \mathbb{R}$ and let $\mathcal{U}$ be the usual uniformity on $\mathbb{R}$.

Define f from $\mathbb{R}$ onto $\mathbb{R}$ by $f(x) = 2x$. It is known that $\mathcal{U}$ consists of all sets which contain a set of the form $\{(x,y): |x-y| < \epsilon\}$ for some 0. Thus by Theorem 2.2 and property (e) of a uniformity, $\mathcal{U}(f)$ consists of all sets which contain a set of the form $f_2\{(x,y): |x-y| < \epsilon\}$.

It is now shown that $\mathcal{U}$ is contained in $\mathcal{U}(f)$. Let U be in $\mathcal{U}$; then there is a set $B = \{(x,y): |x-y| < \epsilon\}$ such that $B \subset U$. Since $\{(x,y): |x-y| < \epsilon/2\}$ is in U, $f_2\{(x,y): |x-y| < \epsilon/2\}$ is in $\mathcal{U}(f)$. But $f_2\{(x,y): |x-y| < \epsilon/2\} = \{(u,v): |u-v| < \epsilon\} = B$. Thus B is in $\mathcal{U}(f)$ and it follows that U is in $\mathcal{U}(f)$.

<u>Example 2.4</u>: Let $X = \mathbb{R}^+$ and let $\mathcal{U}$ be the usual uniformity restricted to $\mathbb{R}^+$. Define f from $\mathbb{R}^+$ onto $\mathbb{R}^+$ by $f(x) = x^2$. Again by Theorem 2.2 and property (e) of a uniformity $\mathcal{U}(f)$ consists of all sets which contain a set of the form $f_2\{(x,y): |x-y| < \epsilon\}$, for some $\epsilon > 0$. However, in this case it is shown that $\mathcal{U} \nsubseteq (f)$. First observe that $f_2\{(x,y): |x-y| < \epsilon\} = \{(x^2,y^2): |x-y| < \epsilon\} = \{(u,v): |\sqrt{u} - \sqrt{v}| < \epsilon\}$
Now if $\mathcal{U} \subseteq (f)$ then $\{(x,y): |x-y| < 1\}$ is in $\mathcal{U}(f)$, hence there is an $\bar{\epsilon} > 0$ such that $\{(u,v): |\sqrt{u} - \sqrt{v}| < \bar{\epsilon}\}$ is contained in $\{(x,y): |x-y| < 1\}$. By the Archemedian law for the reals there is a $\bar{x} \in \mathbb{R}^+$ such that $\sqrt{\bar{x}}\,\bar{\epsilon} > 1$. Let $\sqrt{\bar{y}} = \sqrt{\bar{x}} + \bar{\epsilon}/2$, then $|\sqrt{\bar{x}} - \sqrt{\bar{y}}| = |\sqrt{\bar{x}} - \sqrt{\bar{x}} - \bar{\epsilon}/2| = \bar{\epsilon}/2$. Hence $(\bar{x},\bar{y})$ is in $\{(u,v): |\sqrt{u} - \sqrt{v}| < \bar{\epsilon}\}$, and therefore in $\{(x,y): |x-y| < 1\}$. But $|\bar{x} - \bar{y}| = |\bar{x} - (\sqrt{\bar{x}} + \bar{\epsilon}/2)^2| = |\sqrt{\bar{x}}\,\bar{\epsilon} + \bar{\epsilon}/4| > 1$. Thus $(\bar{x},\bar{y})$ is not in $\{(x,y): |x-y| < 1\}$, and it follows that $\mathcal{U}$ is not contained in $\mathcal{U}(f)$.

Notice that in Example 1.3 $f(x) = 2x$ is a uniformly continuous function, whereas in Example 1.4 $f(x) = x^2$ is not. This turns out to be an important fact. That is, in the next chapter it is shown in a general

setting that f is uniformly continuous if and only if $\mathcal{U}$ is contained in $\mathcal{U}(f)$.

The chapter is concluded with the following transitive property of quotient uniformities.

Lemma 2.3: Let f be a function from X to Y and let g be a function from Y to Z; then the induced functions f_2 and g_2 satisfy the equality $f_2 \circ g_2 = (f \circ g)_2$.

Proof: $(g \circ f)_2\,(x,y) = (g \circ f\,(x),\ g \circ f\,(y)) = (g(f(x)),\ g(f(y)))$
$= g_2(f(x), f(y)) = g_2(f_2(x,y)) = g_2 \circ f_2(x,y)$.

Theorem 2.3: Suppose f maps the uniform space $(X, \mathcal{U})$ onto Y and g maps the uniform space $(Y, \mathcal{U}(f))$ onto Z then $\mathcal{U}[g \circ f] = \mathcal{U}(f)[g]$.

Proof: Let V be in $\mathcal{U}[g \circ f]$; then $(g \circ f)_2^{-1}(V) = f_2^{-1}(g_2^{-1}(V))$ is in $\mathcal{U}$. Thus $g_2^{-1}(V)$ is in $\mathcal{U}(f)$ and it follows that V is in $\mathcal{U}(f)[g]$. Conversely let V be in $\mathcal{U}(f)[g]$; then $g_2^{-1}(V)$ is in $\mathcal{U}(f)$, thus $f_2^{-1}(g_2^{-1}(V))$ is in $\mathcal{U}$, but $f_2^{-1}(g_2^{-1}(V)) = (g_2 \circ f_2)^{-1}(V) = (g \circ f)_2^{-1}(V)$. Thus V is in $\mathcal{U}[g \circ f]$.

III. PROPERTIES OF THE QUOTIENT UNIFORMITY

As has been stated previously the quotient topology is the largest topology which makes f a continuous function. A similar result holds for quotient uniformities. However, before proceeding the concept of uniform continuity will be needed.

Definition 3.1: Let f be a function from a uniform space $(X,\mathcal{U})$ to a uniform space $(Y,\mathcal{V})$ then f is uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$ if and only if for each V in $\mathcal{V}$ the set $\{(x,y): (f(x),f(y)) \in V\}$ is in $\mathcal{U}$.

It will be helpful to have the following equivilent form of uniform continuity.

Theorem 3.1: A function f from $(X,\mathcal{U})$ into $(Y,\mathcal{V})$ is uniformly continuous if and only if for each V in $\mathcal{V}$, there is a U in $\mathcal{U}$ such that $f_2(U)$ is contained in V.

Proof: Suppose f is uniformly continuous and let V be in $\mathcal{V}$. $U = \{(x,y):(f(x), f(y)) \in V\}$ is in $\mathcal{U}$ and it follows that $f_2(U)$ is contained in V. Conversely suppose for each V in $\mathcal{V}$ there is a U in $\mathcal{U}$ such that $f_2(U)$ is contained in V. Let (x,y) be in U; then (f(x),f(y)) is in $f_2(U)$ which is contained in V, hence (f(x),f(y)) is in V. This implies that (x,y) is in $\{(x,y):(f(x),f(y)) \in V\}$. Thus U is contained in $\{(x,y):(f(x),f(y)) \in V\}$, and it follows by property (e) of a uniformity that $\{(x,y):(f(x), f(y)) \in V\}$ is in $\mathcal{U}$.

The relationship between the quotient uniformity and uniform continuity of the function can now be stated.

Theorem 3.2: A function f from a uniform space $(X,\mathcal{U})$ onto a uniform space $(Y,\mathcal{V})$ is uniformly continuous if and only if $\mathcal{V}$ is contained in $\mathcal{U}(f)$, where $\mathcal{U}(f)$ is the quotient uniformity with respect to f.

Proof: Let f map $(X,\mathcal{U})$ onto $(Y,\mathcal{V})$. Suppose $\mathcal{V}\subseteq\mathcal{U}(f)$, and let V be in $\mathcal{V}$. Then $f_2^{-1}(V)$ is in $\mathcal{U}$ by definition of the quotient uniformity. Further, $f_2(f_2^{-1}(V))$ is contained in V, hence f is uniformly continuous by Theorem 3.1. Conversely assume f is uniformly continuous. Let $V\in\mathcal{V}$; then there is a U in $\mathcal{U}$ such that $f_2(U)\subset V$, thus $f_2^{-1}(V)\supset f_2^{-1}(f_2(U))\supset U$. Hence $f_2^{-1}(V)$ is in $\mathcal{U}$ by property (e) of a uniformity and it follows that V is in $\mathcal{U}(f)$. Thus $\mathcal{V}\subseteq\mathcal{U}(f)$ and the proof is complete.

Before proceeding, a fact concerning uniformly continuous functions is proved. This fact is used in the theorem to follow.

Lemma 3.1: Let f map $(X,\mathcal{U})$ onto $(Y,\mathcal{V})$ and let g map $(Y,\mathcal{V})$ onto $(Z,\mathcal{W})$ such that f and g are uniformly continuous. Then the composition $g\circ f$ mapping $(X,\mathcal{U})$ onto $(Z,\mathcal{W})$ is uniformly continuous.

Proof: Let $W\in\mathcal{W}$; since g is uniformly continuous there is a V in $\mathcal{V}$ such that $g_2(V)\subset W$. Further, since f is uniformly continuous there is a U in $\mathcal{U}$ such that $f_2(U)\subset V$. Thus $g_2\circ f_2(U)\subset W$, and by Lemma 2.1 $(g\circ f)_2(U)\subset W$, hence $g\circ f$ is uniformly continuous.

The next theorem is also true in a topological setting with uniform continuity replaced by continuity and uniformities replaced by topologies (See Kelley p. 95).

Theorem 3.3: Let f map $(X,\mathcal{U})$ onto Y, and equip Y with the quotient uniformity $\mathcal{U}(f)$. Then a function g mapping $(Y,\mathcal{U}(f))$ onto $(Z,\mathcal{V})$

is uniformly continuous if and only if $g \circ f$ is uniformly continuous.

Proof: If g is uniformly continuous, the uniform continuity of $g \circ f$ follows from Theorem 3.2 and Lemma 3.1. Conversely assume $g \circ f$ is uniformly continuous, and let V be in $\mathcal{V}$; then there is a U in $\mathcal{U}$ such that $(g \circ f)_2(U) = g_2 \circ f_2(U) = g_2(f_2(U)) \subset V$. But $f_2(U)$ is in $\mathcal{U}(f)$ by Theorem 2.2, hence g is uniformly continuous by Theorem 3.1.

Corollary: Suppose f maps $(X, \mathcal{U})$ onto $(Y, \mathcal{U}(f))$ such that f is one-to-one. Then f^{-1} is uniformly continuous.

Proof: Let g of Theorem 2.3 equal f^{-1}; then $g \circ f = f^{-1} \circ f = i$, which is clearly uniformly continuous. Hence by Theorem 2.3 f^{-1} is uniformly continuous.

The following definition will allow a clearer statement of these ideas.

Definition 2.2: Let f map $(X, \mathcal{U})$ into $(Y, \mathcal{V})$; then f is said to be a uniform isomophism if and only if f is one-to-one, onto, uniformly continuous and has a uniformly continuous inverse. Two spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are said to be uniformly isomophic if and only if there exists a uniform isomophism f from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$.

Theorem 3.4: Suppose f maps $(X, \mathcal{U})$ onto $(Y, \mathcal{U}(f))$. Then f is a uniform isomorphism if and only if f is one-to-one.

Proof: Suppose f is one-to-one; then by assumption f is onto, and f, f^{-1} are uniformly continuous by Theorem 3.2 and the corollary to Theorem 3.3 respectively. Hence f is a uniform isomorphism. The converse is clear.

In a topological setting if f maps $(X, \mathcal{T}_1)$ into $(Y, \mathcal{T}_2)$ such that f is both onto and open, then $\mathcal{T}_2$ is the quotient topology. With the concept of a uniformly open map a similar result can be proved for uniform spaces.

<u>Definition 3.3</u>: Let f map $(X, \mathcal{U})$ into $(Y, \mathcal{V})$; then f is said to be <u>uniformly open</u> if and only if for all U in $\mathcal{U}$ there is a V in $\mathcal{V}$ such that $V[f(x)]$ is contained in $f(U[x])$ for each x in X.

<u>Lemma 3.2</u>: Let f map X onto Y and let $V \subset Y \times Y$ and $U \subset X \times X$. If $V[f(x)] \subset f(U[x])$ for all x in X, then $V \subset f_2(U)$.

<u>Proof</u>: Let (y_1, y_2) be in V; then there is an x_1 in X such that $f(x_1) = y_1$. Thus y_2 is in $V[f(x_1)]$ which implies that y_2 is in $f(U[x_1])$. Hence there is an x_2 in X such that $f(x_2) = y_2$ and x_2 is in $U[x_2]$. Thus (x_1, x_2) is in U, and it follows that (y_1, y_2) is in $f_2(U)$.

<u>Theorem 3.5</u>: Let f map $(X, \mathcal{U})$ into $(Y, \mathcal{V})$; then $\mathcal{V}$ is the quotient uniformity if f is onto, uniformly continuous and uniformly open.

<u>Proof</u>: Since f is onto and uniformly continuous it follows from Theorem 3.2 that $\mathcal{V} \subseteq \mathcal{U}(f)$. Let $V \in \mathcal{U}(f)$; then $f_2^{-1}(V)$ is in $\mathcal{U}$. Now, there is a V_1 in such that $V_1[f(x)] \subset f_2^{-1}(V)[x]$ for all x in X since f is uniformly open. By Lemma 3.2 $V_1 \subset f_2(f_2^{-1}(V)) = V$, and it follows that V is in $\mathcal{V}$ by property (e) of a uniformity.

If $(X, \mathcal{T})$ is a topological space which satisfies the second axiom of countability, it is not true in general that the quotient space $(Y, \mathcal{T}(f))$ also satisfies this axiom. (See Cullen p. 83). However, for uniform spaces the situation is quite different. Before proceeding the notion of a base for a uniformity is made precise.

Definition 3.4: A sub-family $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ if and only if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.

Theorem 3.6: Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{B}$ be a base for $\mathcal{U}$. If f maps X onto Y then the collection $\{f_2(B): B\in\mathcal{B}\}$ is a base for the uniform space $(Y, \mathcal{U}(f))$.

Proof: By Theorem 2.2 $\{f_2(B): B\in\mathcal{B}\}$ is a sub-family of $\mathcal{U}(f)$. Let V be in $\mathcal{U}(f)$; then $f_2^{-1}(V)$ is in $\mathcal{U}$, hence there is a B in $\mathcal{B}$ such that $B\subset f_2^{-1}(V)$. Thus $f_2(B)\subset f_2(f_2^{-1}(V)) = V$, and it follows that $\{f_2(B) : B\in\mathcal{B}\}$ is a base for $(Y, \mathcal{U}(f))$.

Corollary 1: If $(X,\mathcal{U})$ has a countable base then $(Y, \mathcal{U}(f))$ also has a countable base.

Proof: If $\mathcal{B}$ is a countable set then $\{f_2(B) : B\in\mathcal{B}\}$ is also countable.

The metrization theorem for uniform spaces states that a uniform space has a countable base if and only if its uniformity is pseudo-metrizable (i.e., the uniformity is generated by a pseudo-metric). For a proof of this theorem see Kelley p. 186.

This characterization provides another corollary.

Corollary 2: If $(X,\mathcal{U})$ is pseudo-metrizable then $(Y, \mathcal{U}(f))$ is pseudo-metrizable.

Proof: This result follows from corollary 1 and the foregoing remarks.

Given any uniform space $(X,\mathcal{U})$ it is possible to define a topological structure on X in the following way. Let $\mathcal{T}_{\mathcal{U}}$ be the collection of all $G\subset X$ such that there is a U in $\mathcal{U}$ with the property that $U[x]\subset G$. One can

easily verify that this collection is indeed a topology on X. Because of its dependence on $\mathcal{U}$ we call $\mathcal{T}_{\mathcal{U}}$ the <u>topology of the uniformity</u> or simply the <u>uniform topology</u>.

If $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on X, then the family of all sets B[x] for B in $\mathcal{B}$ forms a base for the neighborhood system of x with respect to the uniform topology. Consequently, if $\mathcal{U}$ has a countable base then $\mathcal{T}_{\mathcal{U}}$ satisfies the first axiom of countability. In view of the proceeding theorem we also have the following.

<u>Theorem 3.7</u>: Let $(X,\mathcal{U})$ have a countable base and suppose f maps X onto Y. Then the topological space $(Y, \mathcal{T}_{\mathcal{U}(f)})$ satisfies the first axiom of countability.

<u>Proof</u>: Since $(X,\mathcal{U})$ has a countable base, so does $(Y, \mathcal{U}(f))$ by the corollary to Theorem 3.6. Hence $(Y, \mathcal{T}_{\mathcal{U}(f)})$ satisfies the first axiom of countability by the preceeding remarks.

Next, three important properties of uniform spaces are defined, and the extent to which these properties are possessed by the quotient uniformity is discussed.

<u>Definition 3.5</u>: A uniform space $(X,\mathcal{U})$ is said to be <u>Hausdorff</u> or <u>seperated</u> if and only if $\bigcap\{U: U \in \mathcal{U}\} = \Delta(X)$.

<u>Definition 3.6</u>: A uniform space $(X,\mathcal{U})$ is called <u>totally bounded</u> or <u>pre-compact</u> if and only if for each U in $\mathcal{U}$ there exists a finite set $F \subset X$ such that $U[F] = X$.

In order to define the notion of completeness in a uniform space the following preliminary definition is needed.

<u>Definition 3.7</u>: A <u>net</u> $\{S_n : n \in D\}$ where D is a directed set is

a Cauchy net in the uniform space $(X,\mathcal{U})$ if and only if for each member U of $\mathcal{U}$ (S_m,S_n) is in U whenever $m,n \geq N$ for some N in D.

Definition 3.8: A uniform space $(X,\mathcal{U})$ is called complete if and only if each Cauchy net in the space converges to a point of the space.

If $(X,\mathcal{U})$ is a uniform space such that $\mathcal{U}$ is a seperated uniformity it is not in general true that $(Y,\mathcal{U}(f))$ is also seperated. The fact that $\bigcap\{U : U \in \mathcal{U}\} = \Delta(X)$ implies that $f_2(\bigcap\{U: U \in \mathcal{U}\}) = \Delta(Y)$, but this is not in general equal to $\bigcap\{f_2(U): U \in \mathcal{U}\}$.

Similarly if $(X,\mathcal{U})$ is complete it is not true in general that $(Y, \mathcal{U}(f))$ is complete. However, if f is one-to-one and $(X,\mathcal{U})$ is complete, then $(Y, \mathcal{U}(f))$ is complete. This follows from Theorem 3.4 and the fact that completeness is a uniform invariant. (See Thron p. 76).

However, if $(X,\mathcal{U})$ is pre-compact then it is always true that $(Y, \mathcal{U}(f))$ is pre-compact. By construction f is uniformly continuous and the uniformly continuous image of a pre-compact space is pre-compact. (See Thron p. 76).

Finally, for the material which is in Chapter 4, it is worthwhile to be aware of the following characterization of a complete and totally bounded uniform space.

Theorem 3.7: The uniform space $(X,\mathcal{U})$ is pre-compact and totally bounded if and only if the topological space $(X,\hat{\mathcal{J}}_{\mathcal{U}})$, where $\hat{\mathcal{J}}_{\mathcal{U}}$ is the uniform topology, is compact.

Proof: See Kelley p. 198.

IV. QUOTIENT SPACES

So far we have established that given any uniform space $(X,\mathcal{U})$ and an onto function f from $(X,\mathcal{U})$ to a set Y, it is possible to define a uniformity on Y, denoted by $\mathcal{U}(f)$. It turns out, as the following sequence of theorems will show, that the space Y is really superfluous to the discussion. That is, it is "equivilent" to a set of equivalence classes in X, defined by f.

Theorem 4.1: Let f be a map from $(X,\mathcal{U})$ onto $(Y,\mathcal{U}(f))$ and let $\mathcal{D} = \{f^{-1}(y);\ y \in Y\}$. Define a function $\bar{P}$ from $(X,\mathcal{U})$ to $(\mathcal{D},\mathcal{U}(\bar{P}))$ by $\bar{P}(x) = f^{-1}(f(x))$. Then there is a uniform isomophism g from $(Y,\mathcal{U}(f))$ to $(\mathcal{D},\mathcal{U}(P))$ which makes the following diagram commute.

$$\begin{tikzcd} (X,\mathcal{U}) \arrow[r, "f"] \arrow[d, "\bar{P}"] & (Y,\mathcal{U}(f)) \arrow[dl, dashed, leftrightarrow, "g"] \\ (\mathcal{D},\mathcal{U}(\bar{P})) & \end{tikzcd}$$

Proof: Let $g(y) = f^{-1}(y)$. Now $g \circ f = g(f(x)) = f^{-1}(f(x)) = \bar{P}(x)$. Similarly $g^{-1} \circ \bar{P} = g^{-1}(f^{-1}(f(x)) = f(f^{-1}(f(x)) = f(x)$, thus the diagram commutes. By Theorem 3.1 $\bar{P} = g \circ f$ is uniformly continuous, thus by Theorem 3.3 g is uniformly continuous. Similarly $f = g^{-1} \circ \bar{P}$ is uniformly continuous. It is clear that g is both 1-1 and onto. Thus g is a uniform isomorphism and the proof is complete.

Theorem 4.2: The relation $x \langle f \rangle y$ if and only if $f(x) = f(y)$ is an equivalence relation on X and the corresponding equivalence classes are the sets $\{f^{-1}(y):\ y \in Y\} = \mathcal{D}$.

Proof: It is clear that $\langle f\rangle$ is reflexive, symmetric, and transitive. It remains to show that $\langle f\rangle [x] = \{f^{-1}(f(x)\}$. Let z be in $\langle f\rangle[x]$; then $f(x) = f(z)$, thus z is in $\{f^{-1}(f(x))\}$. Similarly if z is in $\{f^{-1}(f(x))\}$, then $f(z) = f(x)$ and z is in $\langle f\rangle[x]$.

Now let R be any equivalence relation on X and let R[x] denote the corresponding equivalence classes. Let X/R denote $\{R[x]: x \in X\}$, then define P from $(X,\mathcal{U})$ onto $(X/R,\mathcal{U}(P))$ by $P(x) = R[x]$. The function P is called the projection map from $(X,\mathcal{U})$ to the quotient space X/R.

Theorem 4.3: Let f map $(X,\mathcal{U})$ onto $(Y,\mathcal{U}(f))$ then the uniform space $(Y,\mathcal{U}(f))$ is uniformly isomorphic to $(X/\langle f\rangle, \mathcal{U}(P))$.

Proof: By Theorem 4.1 $(Y,\mathcal{U}(f))$ is uniformly isomophic to $(\mathcal{D},\mathcal{U}(\bar{P}))$. By Theorem 4.2 $X/\langle f\rangle = \mathcal{D}$ and $\bar{P}(x) = f^{-1}(f(x)) = \langle f\rangle[x] = P(x)$. Thus, $(Y,\mathcal{U}(f))$ isuniformly isomorphic to $(X/\langle f\rangle, \mathcal{U}(P))$.

V. UNIFORM TOPOLOGIES

As has been stated previously, for each uniform space $(X,\mathcal{U})$ there is an associated topological space $(X,\mathcal{T}_{\mathcal{U}})$ where $\mathcal{T}_{\mathcal{U}}$ is the uniform topology.

Now suppose f is an onto map from the uniform space $(X,\mathcal{U})$ to the uniform space $(Y,\mathcal{U}(f))$. Then the uniformities $\mathcal{U}$ and $\mathcal{U}(f)$ each have an associated uniform topology. Denote these by $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{U}(f)}$ respectively.

$$\begin{tikzcd}
f: (X,\mathcal{U}) \arrow[r, twoheadrightarrow] \arrow[d] & (Y, \mathcal{U}(f)) \arrow[d] \\
f: (X,\mathcal{T}_{\mathcal{U}}) \arrow[r, twoheadrightarrow] & (Y, \mathcal{T}_{\mathcal{U}(f)})
\end{tikzcd}$$

We will call $\mathcal{T}_{\mathcal{U}}$ the "uniform topology" and $\mathcal{T}_{\mathcal{U}(f)}$ "the topology of the quotient uniformity."

As was mentioned in Chapter I, given any topological space $(X,\mathcal{T})$ and a function g from X onto Y it is possible to define a quotient topology $\mathcal{T}(g)$ on Y. Hence, using the function f and the topological space $(X, \mathcal{T}_{\mathcal{U}})$ of the preceeding paragraph we may construct the quotient space $(Y, \mathcal{T}_{\mathcal{U}}(f))$.

At this point, it is natural to ask what relationships exist between $\mathcal{T}_{\mathcal{U}}(f)$ and $\mathcal{T}_{\mathcal{U}(f)}$. It is the aim of this chapter to give a partial answer to this question.

<u>Definition 5.1</u>: A topological space $(X,\mathcal{U})$ is <u>uniformizable</u> if and only if $\mathcal{U}$ is the uniform topology for some uniformity for X.

It is a fact that a topological space is uniformizable if and only if the space is completely regular (see Kelley p. 188). With this

characterization we give a necessary condition for the quotient topology $\mathcal{T}_{\mathcal{U}}(f)$ and the topology of the quotient uniformity $\mathcal{T}_{\mathcal{U}(f)}$ to be identical.

<u>Theorem 5.1</u>: Let $(X,\mathcal{U})$ be a uniform space and let f be a function from X onto an arbitrary set Y. If $\mathcal{T}_{\mathcal{U}}(f) = \mathcal{T}_{\mathcal{U}(f)}$ then the topological space $(Y, \mathcal{T}_{\mathcal{U}}(f))$ is completely regular.

<u>Proof</u>: We know $(Y, \mathcal{T}_{\mathcal{U}(f)})$ is completely regular since $\mathcal{T}_{\mathcal{U}(f)}$ is the uniform topology of the uniformity $\mathcal{U}(f)$. By assumption $(Y, \mathcal{T}_{\mathcal{U}}(f)) = (Y, \mathcal{T}_{\mathcal{U}(f)})$ hence $(Y, \mathcal{T}_{\mathcal{U}}(f))$ is completely regular.

As one might expect, uniform continuity implies continuity, and uniformly open implies open with respect to the uniform topologies.

<u>Theorem 5.2</u>: Let f be a uniformly continuous function from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$ then f is continuous from $(X,\mathcal{T}_{\mathcal{U}})$ to $(Y,\mathcal{T}_{\mathcal{V}})$.

<u>Proof</u>: Let G be in $\mathcal{T}_{\mathcal{V}}$. It must be shown that $f^{-1}(G)$ is in $\mathcal{T}_{\mathcal{U}}$. This is true if and only if, for all x in $f^{-1}(G)$ there is a U in $\mathcal{U}$ such that $U[x] \subset f^{-1}(G)$. Let x be in $f^{-1}(G)$; then f(x) is in G, and there is a V in $\mathcal{V}$ such that $V[f(x)] \subset G$. By uniform continuity there is a U in $\mathcal{U}$ such that $f_2(U) \subset V$, hence $f_2(U)[f(x)] \subset G$. It now follows that $U[x] \subset f^{-1}(f_2(U)[f(x)]) \subset f^{-1}(G)$ and so $G \in \mathcal{T}_{\mathcal{U}}$.

<u>Theorem 5.3</u>: Suppose f is uniformly open from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$ then f is an open map from $(X, \mathcal{T}_{\mathcal{U}})$ to $(Y,\mathcal{T}_{\mathcal{V}})$.

<u>Proof</u>: Let $G \in \mathcal{T}_{\mathcal{U}}$; then for all x in G there is a U in $\mathcal{U}$ such that $U[x] \subset G$. Thus $f(U[x]) \subset f(G)$, and since f is uniformly open there is a V in $\mathcal{V}$ such that $V[f(x)] \subset f(f[x]) \subset f(G)$. Hence f(G) is in $\mathcal{T}_{\mathcal{V}}$.

The following general result, concerning the topologies $\mathcal{T}_{\mathcal{U}(f)}$, and $\mathcal{T}_{\mathcal{U}}(f)$ is a consequence of Theorem 5.2.

Theorem 5.4: Let $(X,\mathcal{U})$ be a uniform space and f a function from X onto Y. Then $\mathcal{T}_{\mathcal{U}(f)} \subset \mathcal{T}_{\mathcal{U}}(f)$.

Proof: The function f from $(X,\mathcal{U})$ to $(Y,\mathcal{U}(f))$ is uniformly continuous by Theorem 3.2, hence f is continuous with respect to the uniform topologies by Theorem 5.2. Now let $G \in \mathcal{T}_{\mathcal{U}(f)}$; then $f^{-1}(G)$ is in $\mathcal{T}_{\mathcal{U}}$ by continuity of f, but this implies that G is in $\mathcal{T}_{\mathcal{U}}(f)$ by definition of the quotient topology.

In general $\mathcal{T}_{\mathcal{U}(f)}$ does not equal $\mathcal{T}_{\mathcal{U}}(f)$, as the following example shows.

Example 5.1: Let $X = \mathbb{R}$ equipped with the usual uniformity. Let $Y = \{a, b\}$ and define f in the following way: $f(x) = a$ for $x < 0$ and $f(x) = b$ for $x < 0$. Then, $Y \times Y = \{(a,a), (a,b), (b,a), (b,b)\}$ and it is easy to see that $\mathcal{U}(f) = \{Y \times Y\}$. It follows that $\mathcal{T}_{\mathcal{U}(f)} = \{\emptyset, \{a,b\}\}$. It is known that $\mathcal{T}_{\mathcal{U}}$ is the usual topology on the real line hence $\mathcal{T}_{\mathcal{U}}(f) = \{\emptyset, \{a, b\}, \{b\}\}$.

Next, three sufficient conditions are given for the topologies $\mathcal{T}_{\mathcal{U}}(f)$ and $\mathcal{T}_{\mathcal{U}(f)}$ to be equal.

Theorem 5.5: Let $(X,\mathcal{U})$ be a uniform space and let f be a function from X onto Y. If f is open with respect to the uniform topologies then $\mathcal{T}_{\mathcal{U}}(f) = \mathcal{T}_{\mathcal{U}(f)}$.

Proof: Let G be in $\mathcal{T}_{\mathcal{U}}(f)$; then $f^{-1}(G)$ is in $\mathcal{T}_{\mathcal{U}}$ and since f is open it follows that $f(f^{-1}(G)) = G$ is in $\mathcal{T}_{\mathcal{U}(f)}$. Hence $\mathcal{T}_{\mathcal{U}}(f) \subseteq \mathcal{T}_{\mathcal{U}(f)}$, and equality follows from Theorem 5.4.

Corollary: If f is uniformly open the $\mathcal{T}_{\mathcal{U}(f)} = \mathcal{T}_{\mathcal{U}}(f)$.

Proof: If f is uniformly open then f is open by Theorem 5.3.

Theorem 5.6: Let $(X,\mathcal{U})$ be a uniform space and let f be a function from X onto Y. If f is closed with respect to the uniform topologies, then $\mathcal{T}_{\mathcal{U}}(f) = \mathcal{T}_{\mathcal{U}(f)}$.

Proof: Let G be in $\mathcal{T}_{\mathcal{U}}(f)$; then $f^{-1}(G)$ is in $\mathcal{T}_{\mathcal{U}}$ and $f(\sim f^{-1}(G)) = \sim f(f^{-1}(G)) = \sim G$ is closed in $(Y, \mathcal{T}_{\mathcal{U}(f)})$ since f is a closed mapping. Hence G is in $\mathcal{T}_{\mathcal{U}(f)}$. Theorem 5.4 gives the reverse inclusion.

Corollary: If $(X,\mathcal{U})$ is compact and $(Y,\mathcal{U}(f))$ is a seperated uniformity then $\mathcal{T}_{\mathcal{U}}(f) = \mathcal{T}_{\mathcal{U}(f)}$.

Proof: A continuous map from a compact space to a Hausdorff space is closed (See Cullen p. 262). Since $(Y, \mathcal{U}(f))$ is seperated it follows that $(Y, \mathcal{T}_{\mathcal{U}(f)})$ is Hausdorff and the conclusion follows.

In order to prove the next theorem it is necessary to review briefly some of the general results on uniform spaces. Given a uniform space $(X,\mathcal{U})$ and an element U in $\mathcal{U}$ it is always possible to find a V in $\mathcal{U}$ such that $V \circ V \subset U$. Furthermore V can be chosen so that it is open in the product topology. This follows since the interior of any member of $\mathcal{U}$ is also a member of $\mathcal{U}$. (See Kelley p. 179). If V is open in the product space $(X\times X, \mathcal{T}\times\mathcal{T})$ then V[x] is open in $(X,\mathcal{T})$ and hence is an open neighborhood of X.

Observe that in Example 4.1 $f^{-1}(b) = (-\infty, 0)$ and $f^{-1}(a) = [0,\infty)$ are not compact sets. This fact turns out to be important.

Theorem 5.7: Let f be a function from the uniform space $(X,\mathcal{U})$ onto the uniform space $(Y,\mathcal{U}(f))$. If $\{f^{-1}(y)\}$ is a compact set for each y in Y then $\mathcal{T}_{\mathcal{U}(f)} = \mathcal{T}_{\mathcal{U}}(f)$.

Proof: By Theorem 5.4 $\mathcal{T}_{\mathcal{U}(f)} \subseteq \mathcal{T}_{\mathcal{U}}(f)$, hence it remains to show that $\mathcal{T}_{\mathcal{U}}(f) \subseteq \mathcal{T}_{\mathcal{U}(f)}$. Let $G \in \mathcal{T}_{\mathcal{U}}(f)$; then $f^{-1}(G)$ is in $\mathcal{T}_{\mathcal{U}}$ and this implies that for all x in $f^{-1}(G)$ there is a U in $\mathcal{U}$ such that $U[x] \subset f^{-1}(G)$. Now G is in $\mathcal{T}_{\mathcal{U}}(f)$ if and only if for all y in G there is a V in $\mathcal{U}(f)$ such that $V[y] \subset G$. Let y be in G, and let $\{f^{-1}(y)\} = C$. Now $C \subset f^{-1}(G)$ hence for all x in C there is a U_x in $\mathcal{U}$ such that $U_x[x] \subset f^{-1}(G)$. By the previous remarks there exists an open V_x in $\mathcal{U}$ such that $V_x \circ V_x \subset U_x$. It follows that $V_x[x]$ is an open neighborhood of x and $\bigcup_{x \in C}(V_x[x])$ is an open cover of C. Let $V_1[x_1] \ldots\ldots V_n[x_n]$ be a finite subcover and let $V = \bigcap_{i=1}^{n} V_i$. It is now shown that $f_2(V)[y] \subset G$. Let z be in $f_2(V)[y]$; then (y,z) is in $f_2(V)$, hence there are elements a, b in X such that f(a) = y, f(b) = z and (a,b) is in V. Now a is in $f^{-1}(y)$ hence a is in C which implies that a is in $V_i[x_i]$ for some i. Also (a,b) is in V so (a,b) is in V_j for all j, j = 1, ...n. In particular (a,b) is in V_i. Now (a,b) in V_i and (x_i,a) in V_i implies that (x_i,b) is in $V_i \circ V_i$ and hence (x_i, b) is in U_i. Thus b is in $U_i[x_i]$ which is contained in $f^{-1}(G)$. It follows that z is in G. Finally $f_2(V)$ is in $\mathcal{U}(f)$ by Theorem 2.2, and the proof is complete.

Corollary: Let $(X,\mathcal{U})$ be compact, $(Y, \mathcal{T}_{\mathcal{U}}(f))$ a T_1 space; then $\mathcal{T}_{\mathcal{U}}(f) = \mathcal{T}_{\mathcal{U}(f)}$.

Proof: Let $y \in Y$; then $\{y\}$ is closed, hence $\{f^{-1}(y)\}$ is closed since f is continuous. Now a closed subset of a compact space is compact hence $\{f^{-1}(y)\}$ is compact.

We note that one could also assume that $(Y, \mathcal{T}_{\mathcal{U}(f)})$, is T_1 in the previous corollary, but since $(Y, \mathcal{T}_{\mathcal{U}(f)})$ is known to be uniformizable and hence completely regular, the space would be $T_{3\frac{1}{2}}$ and thus $(Y, \mathcal{U}(f))$ would necessarily be a seperated uniformity, reducing the result to the corollary to Theorem 5.6.

It is possible to rephrase the previous corollary so that it is independent of the theory of quotient uniformities.

<u>Theorem 5.8</u>: Let $(X, \mathcal{T})$ be a compact, completely regular topological space and suppose f maps X onto Y. If $(Y, \mathcal{T}(f))$ is a T_1 space, then $(Y, \mathcal{T}(f))$ is completely regular.

<u>Proof</u>: Since $(X, \mathcal{T})$ is completely regular there exists a uniformity $\mathcal{U}$ on X such that $\mathcal{T}$ is the uniform topology. It now follows from the corollary to Theorem 5.7 that $\mathcal{T}(f) = \mathcal{T}_{\mathcal{U}(f)}$ and thus $\mathcal{T}(f)$ is uniformizable and hence completely regular.

It is worth noting that the author has been unable to find a proof of this fact independent of the theory developed in this paper.

The fact that continuity implies uniform continuity for a function whose domain is a compact uniform space (See Thron p. 187) allows us to give the following characterization.

<u>Theorem 5.9</u>: Let $(X, \mathcal{U})$ be a compact uniform space and let f be a map from X onto Y. Then $\mathcal{T}_{\mathcal{U}(f)} = \mathcal{T}_{\mathcal{U}}(f)$ if and only if $(Y, \mathcal{T}_{\mathcal{U}}(f))$ is completely regular.

<u>Proof</u>: The necessity follows from Theorem 5.1. For sufficiency suppose $(Y, \mathcal{T}_{\mathcal{U}}(f))$ is completely regular; then it follows that $(Y, \mathcal{T}_{\mathcal{U}}(f))$ is unformizable and hence there is a uniformity $\mathcal{V}$ such that $\mathcal{T}_{\mathcal{V}} = \mathcal{T}_{\mathcal{U}(f)}$. The map f from $(X, \mathcal{T}_{\mathcal{U}})$ to $(Y, \mathcal{T}_{\mathcal{U}}(f))$ is continuous, hence f is uniformly continuous from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$. By Theorem 3.2 this implies that $\mathcal{V} \subseteq \mathcal{U}(f)$ and it follows that $\mathcal{T}_{\mathcal{V}} = \mathcal{T}_{\mathcal{U}}(f) \subseteq \mathcal{T}_{\mathcal{U}(f)}$.

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13. ABSTRACT

In this paper the notion of a quotient topology is extended to uniform spaces, and a quotient uniformity is defined for a uniform space. After the definition is validated its basic properties are investigated and its relation to the quotient topology is discussed.

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